

# Oligopolistic Equilibrium and Financial Constraints\*

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## Abstract

We model a dynamic duopoly in which firms can potentially drive their rivals from the market. A consequence is that, for some ranges of parameters, the static Cournot equilibrium outcome cannot be sustained in an infinitely repeated setting. In those cases, there is a Markov perfect equilibrium in mixed strategies in which one firm will eventually be driven from the market with probability one. The presence of potential bankruptcy makes the set of outcomes supportable via tacit collusion different than in the absence of bankruptcy. We show that producer surplus in the maximum collusive outcome is greater under bankruptcy consideration, since the outcome maximizing joint profits is skewed in favor of more efficient firm. Our numerical simulations illuminate that consumer surplus and social welfare also increase in many cases, although those welfare effects are ambiguous in general.

**Key words:** Financial Constraints, Bankruptcy, Firm Behavior, Dynamic Games.

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## 1. Introduction

There is ample evidence that financial constraints play an important role in the behavior of firms (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997). We begin with the observation that the punishment for the violation of a financial constraint must be severe, otherwise firms would default all the time. Suppose that the punishment is so severe that firms violating financial constraints lose the capacity to compete and disappear (Bolton and Scharfstein, 1990).<sup>1</sup> Firms might then have incentives to take actions that would make it impossible for competitors to fulfill financial constraints in the hope of getting rid of them.

In this paper, we model a quantity-setting duopoly in which firms take into account their and their rivals' financial limits, specifically that firms go bankrupt (exit) if they earn negative profits in a period. We introduce the concept of bankruptcy-free (BF hereafter) outputs. These are output pairs in which each firm's profit is non-negative (so no firm goes bankrupt) and in which no firm could, by changing its output, bankrupt another without bankrupting itself. A critical insight of our analysis is that static Cournot equilibrium outputs can fail to be BF when firms' cost functions are asymmetric. In particular, if firms have constant, but different, average costs, the Cournot outcome can never be BF: a lower-cost firm can bankrupt a higher-cost rival without bankrupting itself by increasing its output to the point that the price falls between their average costs. In a dynamic game, such a move is profitable unless firms discount future too much, because it can get rid of a competitor.

In our dynamic game, in which a quantity-setting game is played for infinitely many periods unless no firm goes bankrupt, the unique Markov Perfect Equilibrium (MPE) in pure strategies, if it exists, is the Cournot equilibrium. But if the Cournot outcome is not BF and firms have incentives to predate for some discounts factors, MPE must entail mixed strategies. *Inter alia*, this suggests that the commonly used constant-marginal-cost Cournot model could be misleading if firms have different marginal costs and are financially constrained.

We show that the mixed strategy MPE exists. Assuming constant average costs and concave profit functions we characterize the equilibrium. The support of each firm's mixed strategy contains exactly one interval. The support of the inferior firm has a mass point in the upper extreme of the

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<sup>1</sup>Even though firms can be reorganized after bankruptcy and continue business, the survival rate of firms after bankruptcy is typically low, 18% US, 20% in UK and 6% in France, see Couwenberg (2001).

interval which coincides with the best reply in the static game to the superior firm's mixed strategy. For the superior firm, the support also contains an isolated mass point which coincides with the best reply in the static game to the inferior firm's mixed strategy. The mass point lies strictly below the interval support. Since the superior firm would not produce a larger output than the best reply unless bankruptcy occurs, outputs in the interval support reflect the predatory activities of the superior firm. The inferior firm becomes bankrupt with positive probability in each period. The introduction of financial constraints implies that monopolization (by the efficient firm) will occur with positive probability in each period and thus, it will almost surely occur in the long run. The probability of predation increases with the discount factor. Moreover, any of the outputs chosen by the superior firm are larger than the Cournot output and the outputs of the inferior firm are smaller than its Cournot output.

We consider the consequences of potential bankruptcy on the set of outcomes supportable via tacit collusion. We show that producer surplus in the maximum collusive outcome is greater under bankruptcy consideration than in the absence of bankruptcy. Thus, in this case, bankruptcy considerations help collusion. However, this greater possibility of collusion does not imply a greater social welfare loss. This is because the outcome that maximizes joint profits is skewed in favor of superior firm and this firm produces more efficiently than the inferior firm. When firms have large discount factors, predation pays off for the superior firm and the inferior firm produces very little fearing bankruptcy. Consequently, total output becomes larger under bankruptcy consideration, and thus total welfare also increases.

We end this introduction with a preliminary discussion of the literature (see more on this in the final section). Although a number of papers demonstrate that the financial structure does affect market outcomes in an oligopoly, most previous studies adopt either static or two-stage models. There are at least two exceptions, Spagnolo (2000) and Kawakami and Yoshida (1997). Both papers make use of games with infinite time like ours. The former examines the role of stock options in repeated Cournot games. In his model, unlike standard repeated games, firms do not necessarily maximize average discounted profits because stock options affect managers' incentives. Taking this effect into consideration, Spagnolo (2000) shows that collusion becomes easier to achieve. The latter incorporates a simple exit constraint into the repeated prisoners' dilemma. In their model, each firm must exit from the market no matter how it plays if the rival deviates over certain number of

periods, and hence no output profile can be bankruptcy free. They show that predations inevitably occur when bankruptcy constraints are asymmetric and firms are long-sighted.

Finally, our approach might provide support to the notion that firms may engage in predatory activities when pursuing profit maximization. Standard explanations of this behavior are based on incomplete information (Milgrom and Roberts, 1982), the learning curve (Cabral and Riordan, 1994) or firms playing an attrition game (Roth, 1996). In our model, firms have complete information. Technology is fixed and firms play standard quantity-setting games. Nevertheless, we obtain predation as a competitive equilibrium in mixed strategies. More importantly, both predation and tacit collusion can be derived (as different equilibria) in a single model, which is a completely new result to the best of our knowledge.

## 2. The model

Two firms compete in an infinite number of periods. In each period firms simultaneously choose quantities. Firms produce a homogeneous product. In order to focus on the strategic decisions regarding outputs, we assume that firms cannot accumulate profits. Firms become bankrupt if they suffer losses in a period. A bankrupt firm exits the market (i.e., ‘produces’ zero every period thereafter). When making its quantity decision in a period, each firm knows what any firm has produced in all previous periods and which firms became bankrupt. The formal definitions are given in Section 4. In the rest of this section we present the elements of the game that is played in each period. For simplicity, the time dimension is not considered yet.

We refer to one of the firms as the superior ( $S$ ) and the other as the inferior ( $I$ ). Let  $j \in \{S, I\}$  denote a firm and  $x_j \in \mathbb{R}_+$  the output of firm  $j$ . Let  $C_i(x_i)$  denote the cost function, where  $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an everywhere twice differentiable function and  $C_i(0) = 0$ . Assume that for all output,  $x$ ,  $AC_S(x) \leq AC_I(x)$ , where  $AC_j(\cdot)$  is firm  $j$ ’s average-cost schedule. Assume that average cost is nondecreasing and twice differentiable. Let  $\mathbf{x} = (x_S, x_I)$  denote an output profile, and let  $X = x_S + x_I$  be the aggregate output. Let  $p(X)$  be the inverse demand function assumed to be strictly decreasing in  $X$  for any positive price and twice differentiable. Derivatives are denoted by primes; e.g.,  $p'(X)$  is the slope of the inverse demand at  $X$ , etc. Profits for firm  $i$  are  $\pi_i \equiv p(X)x_i - C_i(x_i)$ , and written as  $\pi_i(\mathbf{x})$  or as  $\pi_i(x_i, x_j)$ . We assume the classical conditions that

guarantee the existence and uniqueness of a Cournot equilibrium namely, for all  $\mathbf{x} = (x_S, x_I)$ ,

$$p''(X)x_i + p'(X) < 0, \text{ for all } i \in \{S, I\}, \quad (2.1)$$

$$p'(X) - C_i''(x_i) < 0, \text{ for all } i \in \{S, I\}. \quad (2.2)$$

These conditions are satisfied if, for example, demand is linear and cost functions are quadratic.

We denote by  $\mathbf{x}^C = (x_S^C, x_I^C)$  the Cournot output profile and by  $\pi_i^C$  firm  $i$ 's profit at the Cournot output profile.

Central to the analysis of our dynamic set up is the concept of bankruptcy-free (BF) output profiles.

As defined in the Introduction, bankruptcy-free (BF) output profiles are those in which no firm makes negative profit and there is no deviation such that either firm can drive the other into bankruptcy without bankrupting itself. Formally,

**Definition 1.** An output profile  $\hat{\mathbf{x}} = (\hat{x}_S, \hat{x}_I)$  is bankruptcy-free (BF) if

- a)  $\pi_i(\hat{\mathbf{x}}) \geq 0$  for both  $i \in \{S, I\}$ ; and
- b)  $\pi_i(\hat{x}_i, x) \geq 0$  for any  $x$  such that  $\pi_j(x, \hat{x}_i) \geq 0$  ( $i \neq j$ ).

Note that if firm  $i$  is required to make some profit  $v_i$  (it could be either positive or negative) to avoid bankruptcy, we can define a new profit function as  $\tilde{\pi}_i(\mathbf{x}) \equiv \pi_i(\mathbf{x}) - v_i$  and redefine BF with respect to this new profit function.

### 3. Properties of the BF set

In this section we characterize the BF output profiles. The characterizations will become important for the analysis of the dynamic game.

**Lemma 1.** An output profile  $\mathbf{x} = (x_S, x_I)$  is BF if and only if  $\pi_j(\mathbf{x}) \geq 0$  for both  $j$  and

$$AC_j(x_j) \leq AC_k(D(AC_j(x_j)) - x_j) \text{ if } x_j > 0, \quad (3.1)$$

where  $D(\cdot)$  is aggregate demand and  $k \neq j$ .

**Proof.** The definition of BF entails  $\pi_j(\mathbf{x}) \geq 0$  for both  $j$ . If a firm does not produce, it cannot be driven to bankruptcy; hence, assume firm  $j$ 's output is positive and consider whether firm  $k$  can

bankrupt it. Define

$$x(x_j) = \inf\{x \in \mathbb{R}_+ \mid \pi_j(x_j, x) < 0\}. \quad (3.2)$$

By continuity,  $\pi_j(x_j, x(x_j)) = 0$ . It follows that

$$p(x_j + x(x_j)) = AC_j(x_j); \text{ hence, } x(x_j) = D(AC_j(x_j)) - x_j. \quad (3.3)$$

Demand slopes down, so does the price, and thus, a rival's profit decreases with output. Those facts and the assumption that the average cost is non-decreasing mean firm  $k$  can bankrupt  $j$  without bankrupting itself if and only if

$$0 \leq p(x_j + \hat{x}_k) - AC_k(\hat{x}_k) < p(x_j + x(x_j)) - AC_k(x(x_j)) = AC_j(x_j) - AC_k(D(AC_j(x_j)) - x_j) \quad (3.4)$$

for some  $\hat{x}_k > x(x_j)$ . But (3.4) can hold if and only if (3.1) does not. ■

**Corollary 1.** *If the average cost is constant for both firms,  $AC_j(x) = c_j$ ,  $j \in \{S, I\}$ , and  $c_S < c_I$  no output profile with both firms active is BF. In a BF output profile only the superior firm is producing.*

Note that the joint profit maximum is BF because only the superior firm is producing a positive output. The Cournot output is BF if (i)  $c_S = c_I$  or (ii) firms are sufficiently different so that only the superior firm produces in the Cournot equilibrium.

**Proposition 1.** *Let  $\mathbf{x}$  be BF and let  $\mathbf{x}'$  be a smaller profile (i.e.,  $x'_i \leq x_i$  for all  $i \in \{S, I\}$ ). Then,  $\mathbf{x}'$  is also BF.*

**Proof.** We have

$$0 \leq AC_k(D(AC_j(x_j)) - x_j) - AC_j(x_j) \leq AC_k(D(AC_j(x'_j)) - x'_j) - AC_j(x'_j), \quad (3.5)$$

where the first inequality follows from Lemma 1 given the assumption  $\mathbf{x}$  is BF and the second because demand curves slope down and average cost is nondecreasing. Invoking Lemma 1, this chain implies  $\mathbf{x}'$  is BF. ■

A useful characterization of the BF set can be provided under the following additional assumption.

**Assumption 1.** Both firms have an increasing average cost and there exists  $x_S^0$  such that  $AC_I(0) = AC_S(x_S^0) < p(x_S^0)$ .

Observe Assumption 1 holds provided the average cost curve of the superior firm rises above the minimum of the inferior firm's average cost curve and such that the break even output of the superior firm yields a price above the minimum of the inferior firm's average cost. Assumption 1 always holds if, for example, demand is linear and  $C_i(x_i) = \gamma_i x_i^2$  with  $\gamma_i > 0$ .

In the next Lemma we show a consequence of Assumption 1 that is useful to characterize the BF set under Assumption 1.

**Lemma 2.** *Assumption 1 implies that there exists a unique  $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I) \neq (0, 0)$  such that  $\pi_i(\bar{\mathbf{x}}) = 0$  for all  $i \in \{S, I\}$ .*

**Proof.** Since average costs are increasing, the functions  $AC_I(\cdot)$  and  $AC_S(\cdot)$  are invertible, so for any level  $\lambda \in [AC_S(x_S^0), \infty)$  there exist  $x_I(\lambda)$  and  $x_S(\lambda)$ , continuous in  $\lambda$ , such that

$$\lambda = AC_I(x_I(\lambda)) = AC_S(x_S(\lambda)). \quad (3.6)$$

By Assumption 1, if  $\lambda = AC_S(x_S^0)$ , then

$$p(x_I(\lambda) + x_S(\lambda)) > AC_I(x_I(\lambda)) = AC_S(x_S(\lambda)). \quad (3.7)$$

But as  $\lambda \rightarrow \infty$  the left term goes to 0, while the middle and right terms stay above 0; by continuity and given that demand decreases and average costs are increasing, there must exist a unique  $\lambda^*$  such that

$$p(x_I(\lambda^*) + x_S(\lambda^*)) = AC_I(x_I(\lambda^*)) = AC_S(x_S(\lambda^*)). \quad (3.8)$$

Let  $\bar{\mathbf{x}} = (x_I(\lambda^*) + x_S(\lambda^*))$ . By construction,  $\pi_i(\bar{\mathbf{x}}) = 0$  for all  $i \in \{S, I\}$ . ■

**Lemma 3.** *Suppose Assumption 1 holds. Let  $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$  be as described in Lemma 2. Then the set of BF output profiles is:*

$$BF = \{(x_S, x_I) \mid 0 \leq x_i \leq \bar{x}_i \text{ for } i \in \{S, I\}\}. \quad (3.9)$$

**Proof.** Trivially,  $\bar{\mathbf{x}}$  is BF. By Proposition 1, it follows that all  $\mathbf{x} < \bar{\mathbf{x}}$  are BF. To complete the proof, we need to show that no  $\mathbf{x}$  can be BF when  $x_j > \bar{x}_j$  for one or more  $j$ . Because demand slopes down and average costs are increasing, if  $\mathbf{x} > \bar{\mathbf{x}}$ , then  $\pi_i(\mathbf{x}) < \pi_i(\bar{\mathbf{x}}) = 0$  for all  $i \in \{S, I\}$ , so  $\mathbf{x}$  is not BF. Suppose then,  $x_j > \bar{x}_j$ , but  $x_i < \bar{x}_i$ . Observe  $x_j < \bar{x}_j + \bar{x}_i$ ; if not, the chain

$$p(x_i + x_j) - AC_j(x_j) \leq p(\bar{x}_i + \bar{x}_j) - AC_j(x_j) < p(\bar{x}_i + \bar{x}_j) - AC_j(\bar{x}_j) = 0 \quad (3.10)$$

shows  $\mathbf{x}$  is not BF. Given  $x_j < \bar{x}_j + \bar{x}_i$ , there exists  $\hat{x}_i > 0$  such that  $\hat{x}_i + x_j = \bar{x}_j + \bar{x}_i$ . Let

$$p^* = p(\hat{x}_i + x_j) = p(\bar{x}_j + \bar{x}_i). \quad (3.11)$$

The average cost is increasing; hence,

$$p^* - AC_i(\hat{x}_i) > p^* - AC_i(\bar{x}_i) = 0 = p^* - AC_j(\bar{x}_j) > p^* - AC_j(x_j). \quad (3.12)$$

Expression (3.12) entails that firm  $i$  can bankrupt firm  $j$  without bankrupting itself; hence,  $\mathbf{x}$  is not BF. ■

To close this section, we work out in an example the conditions under which joint profit maximum and Cournot's output profiles are BF under Assumption 1.

**Example 1.** Let demand be linear,  $p(x_S + x_I) = a - x_S - x_I$ , and cost be quadratic,  $C_S(x) = \gamma_S x^2$ ,  $C_I(x) = \gamma_I x^2$  with  $0 < \gamma_S \leq \gamma_I$ . By Lemma 3 the BF set is completely characterized by the output profile  $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$  such that  $AC_I(\bar{x}_I) = AC_S(\bar{x}_S) = p(\bar{x}_S + \bar{x}_I)$ , thus,  $\gamma_S \bar{x}_S = \gamma_I \bar{x}_I = a - \bar{x}_S - \bar{x}_I$ .

$$\bar{x}_S = \frac{\gamma_I a}{(1 + \gamma_S)(1 + \gamma_I) - 1}; \quad \bar{x}_I = \frac{\gamma_S a}{(1 + \gamma_S)(1 + \gamma_I) - 1}. \quad (3.13)$$

The Cournot equilibrium is given by

$$x_S^C = \frac{(1 + 2\gamma_I)a}{4(1 + \gamma_I)(1 + \gamma_S) - 1} : \quad (3.14)$$

$$x_I^C = \frac{(1 + 2\gamma_S)a}{4(1 + \gamma_I)(1 + \gamma_S) - 1} \quad (3.15)$$

For the superior firm, it is always the case that  $x_S^C \leq \bar{x}_S$ . For the inferior firm,  $x_I^C \leq \bar{x}_I$  if and only if

$$\gamma_S \geq \frac{\gamma_I}{2(1 + \gamma_I)}. \quad (3.16)$$

Thus, the Cournot equilibrium is BF if and only if condition (3.16) holds. The right hand side of (3.16) is increasing in  $\gamma_I$ , which is the parameter defining the marginal cost of the inferior firm. Thus the larger  $\gamma_I$  is, i.e. the more inefficient the inferior firm is, the easier it is for the superior firm to bankrupt the inferior firm. Conversely, the larger  $\gamma_S$  is, the more difficult it is for the superior firm to bankrupt the inferior firm.

Finally, in this example, the joint-profit-maximizing output,  $\mathbf{x}^J = (x_S^J, x_I^J)$ , is always BF because the marginal cost of both firms is the same,  $\gamma_S x_S^J = \gamma_I x_I^J$  (production efficiency) and profits are



non-negative for both firms. Furthermore, since  $\gamma_S \bar{x}_S = \gamma_I \bar{x}_I$ , trivially  $x_j^J \leq \bar{x}_j$  for all  $j \in \{S, I\}$ . Following the same reasoning, the perfectly competitive equilibrium, which is also the efficient allocation, is also BF.

#### 4. Dynamic Competition with Bankruptcy

In this section we focus on the dynamic model.

In each period  $t$  each firm  $i \in \{S, I\}$  chooses an output denoted by  $x_i^t$ . Let  $\mathbf{x}^t = (x_I^t, x_S^t)$  be a profile of outputs in period  $t$ . The profits obtained by firm  $i$  in period  $t$  are  $\pi_i(\mathbf{x}^t)$   $t = 0, 1, \dots, \tau, \dots$ . We define a state at  $t$  as the set of firms that did not fall into bankruptcy in previous periods called *active* firms. Let  $\delta \in (0, 1)$  be the common discount factor. Discounted profits for firm  $i$  are  $\Pi_i = \sum_{t=0}^{\infty} \delta^t \pi_i(\mathbf{x}^t)$ . The continuation profit in period  $t$  is given by  $\Pi_i^t = \sum_{r=0}^{\infty} \delta^r \pi_i(\mathbf{x}^{t+r})$ . At period 0 the game begins with the null history  $h^0$ . For  $t \geq 1$ , a history, denoted by  $h^t = (\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^{t-1})$ , is a list of all outputs in all periods before  $t$ . A strategy for firm  $i$ ,  $\sigma_i$ , (pure or mixed) is a sequence of maps, one for each period  $t$ , mapping all possible period  $t$  histories into a probability distribution in outputs. Let  $\sigma = (\sigma_I, \sigma_S)$  denote a strategy profile (pure or mixed). A Markovian strategy for firm  $i$  is a mapping from the set of active firms into a probability distribution on outputs. A Subgame Perfect Nash Equilibrium (SPNE) is a collection of strategies which are a NE in every possible subgame. A Markov perfect equilibrium (MPE) is a SPNE in which firms use Markovian strategies only. To ease notation whenever no confusion can arise, we drop the time superindex.

In infinitely repeated games without bankruptcy considerations there is only one state, and hence the Markovian strategy coincides exactly with the Cournot output. Under bankruptcy considerations, when the Cournot output profile is BF, the unique MPE is that both firms produce the Cournot outcome when both are active and, when only one firm is active, this firm produces the monopoly outcome. In equilibrium both firms are active in every period. A different analysis has to be made when the Cournot outcome profile is not BF because it could be the case that for some discounts factors one firm may have incentives to bankrupt the other firm. In the following Lemma we provide the range of the discount factor that is needed in order to prevent such a deviation.

**Lemma 4.** *Suppose that  $(x_S^C, x_I^C)$  is not BF. Then, there exists  $\underline{\delta} < 1$  such that the Markovian strategy  $x_i = x_i^C, i \in \{S, I\}$  in the states with all firms active, and  $x_i = x_i^M$  in states where only firm  $i$  is active constitutes a MPE if and only if  $\delta \leq \underline{\delta}$ .*

**Proof.** Deviations in states in which only one firm is active are not profitable because the active firm is producing the monopoly outcome and the other firm is out of the market. Thus, only deviations at states with both firms active are possible. Note that if both firms are active and produce Cournot outputs, profits for firm  $i$  are  $\pi_i^C/(1-\delta)$ . Given that the Cournot output profile is not BF, a potential profitable deviation is such that one firm drives the other to bankruptcy without bankrupting itself. The discounted profits for this move are  $\pi_i^D + \delta\pi_i^M/(1-\delta)$ , where  $\pi_i^M$  are monopoly profits and  $\pi_i^D$  are profits in the deviation for firm  $i$ . Firm  $i$  drives firm  $j$  to bankruptcy by producing an outcome  $\hat{x}_i > x(x_j^C)$ , where  $x(x_j^C)$  is such that  $\pi_j(x_j^C, x(x_j^C)) = 0$ . Given that  $\pi_j(x_j^C, x_i^C) \geq 0$ ,  $x(x_j^C) \geq x_i^C$ , and therefore, for all  $\hat{x}_i > x(x_j^C)$ ,  $\pi_i(\hat{x}_i, x_j^C) < \pi_i(x(x_j^C), x_j^C)$ . Thus, driving firm  $j$  to bankruptcy is not a profitable deviation for firm  $i$  if and only if

$$\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M. \quad (4.1)$$

For  $\delta \simeq 0$  the right hand side of 4.1 is approximately  $\pi_i(x(x_j^C), x_j^C)$ , and then the inequality holds because  $\pi_i^C \geq \pi_i(x(x_j^C), x_j^C)$ . For  $\delta \simeq 1$  the right hand side of 4.1 is approximately  $\pi_i^M$ , and the inequality does not hold because  $\pi_i^C < \pi_i^M$ . Since the right hand side of 4.1 is decreasing in  $\delta$ , by the intermediate value theorem there is  $\underline{\delta}_i$  such that  $\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M$  if and only if  $\delta \leq \underline{\delta}_i$ . In conclusion, by taking  $\underline{\delta} = \min\{\underline{\delta}_i\}_{i \in \{S, I\}}$ , we get the result. ■

When  $\delta > \underline{\delta}$ , a MPE may involve mixed strategies. We start by a general observation that will be useful later on.

**Lemma 5.** *For any pure strategy MPE, no firm goes bankrupt.*

**Proof.** Suppose that firm  $i$  goes bankrupt in some period  $t$ , which happens only if its profit in  $t$  is negative. Since the profits after bankruptcy are always zero,  $i$ 's continuation profit at  $t$  is zero. However, producing nothing at  $t$  and in any of the following periods, assures zero profits, so firm  $i$  can profitably deviate by choosing  $x_i^t = 0$  at  $t$ . Thus we derive contradiction. ■

Note that Lemma 5 holds even when strategies are not constrained to be Markovian. The following lemma shows that when the repeated Cournot outcome cannot be a MPE equilibrium, no MPE equilibrium in pure strategies exists when  $\delta$  is large.

**Lemma 6.** *For any  $\delta > \underline{\delta}$ , there is no MPE in pure strategies.*

**Proof.** Given no bankruptcy occurs in an equilibrium in pure strategies (Lemma 5), the repeated Cournot outcome when both firms are active is a unique mutual best reply in pure strategies that are Markovian. However, it cannot be an equilibrium for  $\delta > \underline{\delta}$  by Lemma 4. ■

In light of Lemma 6, we study equilibria in mixed strategies when  $\delta > \underline{\delta}$ . The existence of a MPE is guaranteed by an extension of a theorem proved in Dasgupta and Maskin (1986) which we include in the Appendix.

**Proposition 2.** *For any  $\delta$ , there exists at least one MPE, possibly, in mixed strategies.*

The characterization of the mixed strategy equilibrium is not an easy task. To characterize the equilibrium support, we impose two assumptions.

**Assumption AC** The average cost for each firm is constant.

**Assumption EC** Each firm  $i$ 's expected per-period profit function (given other firm's mixed strategy) is strictly concave in  $x_i$  for any  $x_i \geq 0$ .

Let  $E[\pi_i(x_i, x_j) \mid \sigma_j]$  be the expected per-period profit of firm  $i$  given that firm  $j$  is using the mixed strategy  $\sigma_j$ .

**Proposition 3.** *Under assumptions AC and EC, and  $\delta > \underline{\delta}$ , a mixed strategy MPE must satisfy the following conditions:*

- (i) Firm  $S$  randomizes over  $x_S^0 \cup (\underline{x}_S^*, \overline{x}_S^*]$  where  $x_S^0 < \underline{x}_S^*$ .
- (ii) Firm  $I$  randomizes over  $(\underline{x}_I^*, \overline{x}_I^*) \cup \overline{x}_I^*$ .
- (iii)  $\overline{x}_I^* = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ .
- (iv)  $x_S^0 = \arg \max_{x_S} E[\pi_S(x_I, x_S) \mid \sigma_I]$ .
- (v)  $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$ .
- (vi)  $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$ .
- (vii)  $\pi_I(\overline{x}_I^*, x_S^0) > 0$ .

The proof of Proposition 3 is long and involved and it is shown in the Appendix. We highlight here some of the properties. 1) The support of each firm's mixed strategy contains exactly one

interval and one mass point. The mass point for the superior firm is an isolated point and the mass point for the inferior firm is the superior extreme of the interval. Moreover, for each firm the mass point coincides with the best reply in the static game to the other firm's mixed strategy. 2) Given condition (vi), by producing  $x_S^0$  the superior firm cannot bankrupt firm  $I$ . Thus, this firm will not produce a larger output than  $x_S^0$  unless bankruptcy occurs. Therefore, outputs in  $(\underline{x}_S^*, \overline{x}_S^*]$  reflect the predatory activities of the superior firm. 3) For the inferior firm, the length of its support reflects the delicate balance between a larger output and a larger probability of bankruptcy, which occurs with positive probability in each period. The inferior firm will eventually be driven from the market with probability one.

In the following proposition we give the characterization of the probability distribution on the support for each firm under the assumption that demand is linear.

**Proposition 4.** *Suppose that AC and EC holds and demand is linear,  $p(X) = a - X$ . Let  $c_I$  and  $c_S$  be the marginal cost of  $I$  and  $S$  respectively, let  $a_I = a - c_I$  and  $a_S = a - c_S$ . For any  $\delta > \underline{\delta}$ , a mixed strategy MPE is such that:*

(i) *Firm  $S$  randomizes according to a mixture cdf  $p_S + (1 - p_S)F_S(x_S)$  with support  $x_S^0 \cup (\underline{x}_S^*, \overline{x}_S^*]$  where  $F_S(x_S)$  is a cdf on  $(\underline{x}_S^*, \overline{x}_S^*]$  and  $p_S$  is the probability of producing  $x_S^0$  with*

$$F'_S(x_S) = \frac{2(x_S - \underline{x}_S^*)}{(\overline{x}_S^* - \underline{x}_S^*)^2}, \text{ and} \quad (4.2)$$

$$p_S = 1 - \frac{(1 - \delta)(\overline{x}_S^* - \underline{x}_S^*)^2}{\delta(\overline{x}_S^* - 2\underline{x}_S^* + a_I)(a_I - \overline{x}_S^*)} \quad (4.3)$$

(ii) *Firm  $I$  randomizes according to a mixture cdf  $p_I + (1 - p_I)G_I(x_I)$  with support  $[\underline{x}_I^*, \overline{x}_I^*) \cup \overline{x}_I^*$  where  $G_I(x_I)$  is a cdf on  $[\underline{x}_I^*, \overline{x}_I^*)$  and  $p_I$  is the probability of producing  $\overline{x}_I^*$  with*

$$G'_I(x_I) = \frac{2(x_S^0 - (a_I - x_I))}{(\overline{x}_I^* - \underline{x}_I^*) \left( \overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0 \right)}, \text{ and} \quad (4.4)$$

$$p_I = 1 - \frac{(\overline{x}_I^* - \underline{x}_I^*) \left( \overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0 \right)}{\delta \left[ (2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M \right]}. \quad (4.5)$$

where  $\pi_S^M = (\frac{a_S}{2})^2$

(iii) Finally,  $x_S^0, \underline{x}_S^*, \overline{x}_S^*, \underline{x}_I^*, \overline{x}_I^*$  are the solutions of the following system:

$$2\underline{x}_S^* - a_I = p_S x_S^0 + (1 - p_S) \int_{\underline{x}_S^*}^{\overline{x}_S^*} x_S F'_S(x_S) dx_S \quad (4.6)$$

$$a_S - 2x_S^0 = p_I \overline{x}_I^* + (1 - p_I) \int_{\underline{x}_I^*}^{\overline{x}_I^*} x_I G'_I(x_I) dx_I \quad (4.7)$$

$$\frac{1}{1 - \delta} (x_S^0)^2 = (2x_S^0 - \overline{x}_S^*) \overline{x}_S^* + \delta \frac{\pi_S^M}{1 - \delta} \quad (4.8)$$

$$\overline{x}_I^* = a_I - \underline{x}_S^* \quad (4.9)$$

$$\underline{x}_I^* = a_I - \overline{x}_S^* \quad (4.10)$$

The proof of Proposition 4 is in the Appendix. We comment here the content of each of the equations that define the support of the mixed strategy for each firm. Equations (4.9) and (4.10) follow from (v) and (vi) in Proposition 3. The right hand side of equation (4.6) is the expected value of the outputs in the support of firm  $S$ 's mixed strategy. This expected value has to be equal to  $2\underline{x}_S^* - a_I$  for two reasons: first, by (iii) in Proposition 3,  $\overline{x}_I^* = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ , since demand is linear,  $\overline{x}_I^*$  is the solution of  $a_I - 2\overline{x}_I^* - Ex_S = 0$ . Secondly, by (vi) in Proposition 3,  $\overline{x}_I^* = a_I - \underline{x}_S^*$ . Thus,  $Ex_S = 2\underline{x}_S^* - a_I$ . The right hand side of equation (4.7) is the expected value of the outputs in the support of firm  $I$ 's mixed strategy. By (iv) in Proposition 3,  $x_S^0 = \arg \max_{x_S} E[\pi_S(x_I, x_S) \mid \sigma_I]$ , since demand is linear,  $x_S^0$  is the solution of  $a_S - 2x_S^0 - Ex_I = 0$ . Thus,  $Ex_I = a_S - 2x_S^0$ . The left hand side of equation (4.8) is the expected payoff of firm  $S$  at  $(x_S^0, \sigma_I)$ . By (vii) in Proposition 3,  $\pi_I(\overline{x}_I^*, x_S^0) > 0$ , thus, when firm  $S$  produces  $x_S^0$ , the probability of bankruptcy for firm  $I$  is zero, and consequently the payoff of firm  $S$  is  $(a_S - x_S^0 - Ex_I)x_S^0/(1 - \delta) = (x_S^0)^2/(1 - \delta)$ . The right hand side of equation (4.8) is the expected payoff of firm  $S$  at  $(\overline{x}_S^*, \sigma_I)$ . Since by (v) in Proposition 3,  $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$ , the probability of bankruptcy for firm  $I$  is 1. Thus, the payoff at  $(\overline{x}_S^*, \sigma_I)$  is  $\pi_S(\overline{x}_S^*, \sigma_I) + \delta \pi_S^M/(1 - \delta)$ . Finally, since both  $x_S^0$  and  $\overline{x}_S^*$  are in the support of  $S$ 's mixed strategy, the payoffs are equal.

Finally, we highlight that when  $\delta \rightarrow \underline{\delta}$ ,  $p_I = 1$ ,  $p_S = 1$ ,  $x_S^0 = x_S^C$ ,  $\overline{x}_I^* = x_I^C$ ,  $\underline{x}_I^* = \overline{x}_I^*$ ,  $\underline{x}_S^* = \overline{x}_S^* = a_I - x_I^C$ , is a solution of the system.

The equations that characterize equilibrium do not allow for an explicit solution, but we have made some simulations for  $a_S = 125$  and  $a_I \in \{100, 105, 110, 115, 120, 124\}$  and  $\delta \simeq \underline{\delta}$  up to .99. We present the results for  $a_S = 125$ ,  $a_I = 100$ . The findings that we show here are robust to the consideration of different  $a_I$ . These simulations are available under request.

**Example 2.** In this example  $a_I = 100$ , and  $a_S = 125$ . In this case,  $\underline{\delta} = 0.3077$ , and the Cournot outputs are  $x_I^C = 25$  and  $x_S^C = 50$ . The table below shows how the support of the mixed strategy changes with  $\delta$ , for  $\delta > \underline{\delta}$ .

$\delta$	$\underline{x}_I^*$	$\overline{x}_I^*$	$x_S^0$	$\underline{x}_S^*$	$\overline{x}_S^*$
0.31	24.86	24.99	50.001	75.001	75.14
0.34	23.1	24.8	50.15	75.2	76.9
0.4	19.48	23.83	50.89	76.17	80.52
0.5	13.66	21.36	52.63	78.64	86.34
0.6	8.24	17.98	54.68	82.02	91.76
0.7	3.71	13.22	56.95	86.78	96.29
0.8	1.06	7.43	59.27	92.57	98.94
0.9	0.17	2.87	61.16	97.13	99.83
0.99	0.0012	0.23	62.39	99.77	99.9988

We see that the length of both supports decreases when  $\delta$  is sufficiently large ( $\delta > 0.7$ ), so the behavior of both firms becomes much more predictable when firms do not discount future very much. As expected, the support for firm  $S$  increases with  $\delta$  and the support for firm  $I$  decreases with  $\delta$ . This reflects that predation is more profitable when firms discount future less. Finally, when  $\delta \rightarrow 1$ , firm  $S$  sets  $x_S^0$  equal to the monopoly output 62.50.

In the following table we show how  $p_I$ , the probability of producing  $\overline{x}_I^*$ , and  $p_S$ , the probability of producing  $x_S^0$  changes with  $\delta$ .

$\delta$	$p_I$	$p_S$
0.31	0.99	0.999
0.34	0.88	0.99
0.4	0.73	0.95
0.5	0.60	0.85
0.6	0.54	0.72
0.7	0.57	0.54
0.8	0.70	0.31
0.9	0.87	0.12
0.999	0.99	0.01

We see that the probability to produce  $x_S^0$  decreases with  $\delta$  showing how the predation outputs in  $(\underline{x}_S^*, \overline{x}_S^*]$  become dominant when firms discount future very little. In contrast, the probability of producing  $\overline{x}_I^*$  is not monotonic on  $\delta$ .

In the following table we show how the expected output changes with  $\delta$ .

$\delta$	$Ex_I$	$Ex_S$
0.31	24.998	50.003
0.34	24.7	50.4
0.4	23.22	52.34
0.5	19.74	57.28
0.6	15.65	64.04
0.7	11.11	73.56
0.8	6.46	85.15
0.9	2.68	94.25
0.99	0.22	99.54

We see that the fear of being predated makes firm  $I$  produce less and less when predation becomes more and more profitable (large  $\delta$ ) which, in turn, makes room for firm  $S$  to produce a larger output.

Next we compute the probability that both firms survive,  $p_1$ .

$\delta$	$p_1$
0.31	0.999
0.34	0.991
0.4	0.95
0.5	0.87
0.6	0.77
0.7	0.61
0.8	0.38
0.9	0.16
0.99	0.01

Again we see that, when  $\delta$  increases, despite the shrinking of firm  $I$  this firm is bankrupt with a larger and larger probability. This reflects the larger profitability of predation. This is displayed

in the next table which shows payoffs in the Markov equilibrium.

$\delta$	$V_I(\sigma_I, \sigma_S)$	$V_S(\sigma_S, \sigma_I)$
0.31	905.68	3623.34
0.34	927.37	3811.28
0.4	914.93	4316.04
0.5	793.95	5540.32
0.6	571.10	7473.90
0.7	281.18	10809.71
0.8	73.39	17564.55
0.9	9.29	37400.32
0.99	0.054	389195.95

(4.11)

Note that for  $\delta = 0.31$  (which is almost the threshold  $\underline{\delta} = 0.3077$ ) the payoffs in the MPE are close to the discounted Cournot payoffs ( $\pi_I^C/(1 - \delta) = 905.79$ , and  $\pi_S^C/(1 - \delta) = 3623.20$ ), and when  $\delta = 0.99$ ,  $V_S(\sigma_S, \sigma_I) \simeq \pi_S^M/(1 - \delta)$ .

We now focus on the consequences of bankruptcy on the maximum collusive outcome comparing the traditional framework without bankruptcy considerations and our framework. For the traditional case, we use Cournot reversion as punishment, formally

$$\begin{aligned} \max \pi_S(x_S, x_I) + \pi_I(x_I, x_S) \\ \text{s.t. } \pi_S(x_S, x_I) &\geq (1 - \delta)\left(\frac{a_S - x_I}{2}\right)^2 + \delta\pi_S^C \\ \pi_I(x_I, x_S) &\geq (1 - \delta)\left(\frac{a_I - x_S}{2}\right)^2 + \delta\pi_I^C \end{aligned}$$

Under bankruptcy considerations we use the MPE profits as punishment and we also have to consider the incentives to predate. Formally,

$$\begin{aligned} \max \pi_S(x_S, x_I) + \pi_I(x_I, x_S) \\ \text{s.t. } \pi_S(x_S, x_I) &\geq (1 - \delta)\left(\frac{a_S - x_I}{2}\right)^2 + \delta(1 - \delta)V_S(\sigma_S, \sigma_I) \\ \pi_I(x_I, x_S) &\geq (1 - \delta)\left(\frac{a_I - x_S}{2}\right)^2 + \delta(1 - \delta)V_I(\sigma_I, \sigma_S) \\ \pi_S(x_S, x_I) &\geq (1 - \delta)(a_S - a_I)(a_I - x_I) + \delta\pi_S^M \end{aligned}$$



As in the previous example, we have computed the maximum collusive outcome for  $a_S = 125$  and  $a_I \in \{100, 105, 110, 115, 120, 124\}$  and  $\delta \simeq \underline{\delta}$  up to .99. We show here the results for  $a_S = 125, a_I = 100$ .

**Example 3.** *Going back to Example 2, the following table shows the maximum joint profits under bankruptcy considerations (left part of the table, named B), and without bankruptcy considerations (right part of the table, named NB).*

<i>B</i>	$\delta$	$\pi_I + \pi_S$	$(x_I, x_S)$	<i>NB</i>	$\delta$	$\pi_I + \pi_S$	$(x_I, x_S)$
	0.31	3375.33	(20.982,44.04)		0.31	3375.28	(20.984,44.039)
	0.34	3389.66	(20.31,43.61)		0.34	3389.66	(20.57,43.44)
	0.4	3437.27	(18.55,46.25)		0.4	3412.59	(19.74,42.24)
	0.5	3549.79	(14.24,48.92)		0.5	3437.5	(18.59,41.89)
	0.6	3667.23	(9.56,52.88)		0.6	3457.6	(17.66,42.17)
	0.7	3775.83	(5.21,57.75)		0.7	3474.68	(16.85,42.44)
	0.8	3845.58	(2.41,60.66)		0.8	3489.47	(16.13,42.69)
	0.9	3881.08	(1.00,61.82)		0.9	3502.45	(15.48,42.93)
	0.99	3903.90	(0.09,62.43)		0.99	3512.88	(14.96,43.13)

We see that when bankruptcy is taken into consideration, joint profits supported by grim trigger strategies are larger than those in the standard case. It is particularly noteworthy that when  $\delta$  is very close to 1 and thus, the maximum profit achievable under non bankruptcy is almost on the profit frontier, the scenario with bankruptcy allows much larger profits to be implemented. This is because the possibility of bankruptcy affects both firms asymmetrically. On the one hand, the superior firm has the chance to become a monopolist. On the other hand, the inferior firm has the risk to be eliminated. Thus, the mixed strategy equilibrium is more favorable to the superior firm and shifts profits in the direction of the superior firm. In Figure 1 we show the maximum profit possibility frontier which is the solid line, and the isoprofits which is the dotted line. It is easy to see that, when we are close to the frontier, a decrease in the profits of the inferior firm increases joint profits as happens in this example.

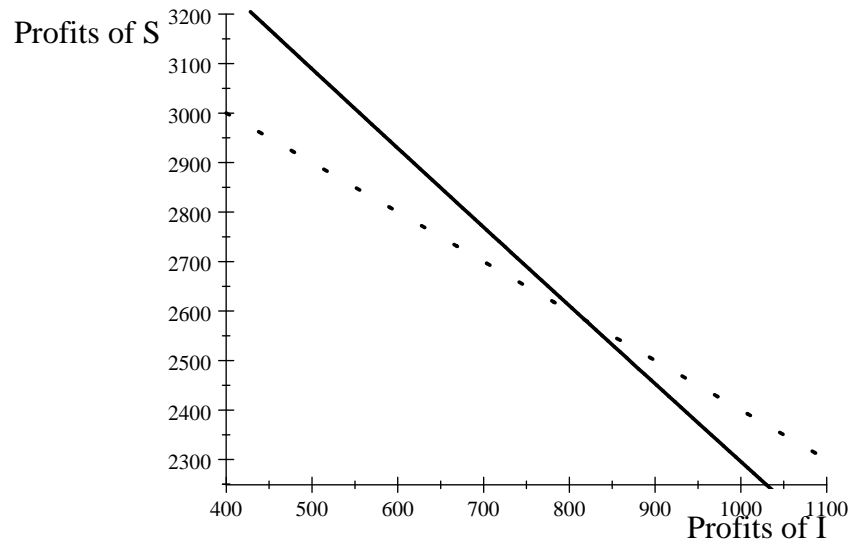


Figure 1

The findings in the previous two examples are robust to the consideration of different  $a_I$ . But in the next example we see that the effect of bankruptcy constraints on social welfare depends on more subtle considerations.

**Example 4.** One period social welfare -denoted by  $W$ - is defined as the sum of consumer ( $aX - X^2/2 - pX$ ) and producer ( $pX - c_S x_S - c_I x_I$ ) surpluses. Thus  $W = a_S x_S + a_I x_I - X^2/2$ . The following table shows how the welfare between the maximum collusive outcome with  $(WB)$  and without  $(WNB)$  bankruptcy varies with  $a_I$  and  $\delta$ . We have selected only the values for which this difference is negative. For the other values in our simulations that we do not show here this difference is positive (the table is available under request)

$a_I$	$\underline{\delta}$	$\delta$	$WB - WNB$	$X^B$	$X^N$
105	0.338	0.34	-0.0043	65.1170	65.1173
105		0.35	-0.21	64.75	64.76
110	0.367	0.37	-0.032	65.188	65.190
110		0, 4	-3.8872	63.938	64.0726
115	0.39	0, 4	-0.1987	65.247	65.252
120	0.42	0.43	-0.8578	65.29	65.31
120		0.45	-7.45	64.36	64.51
124	0.44	0.45	-1.165	65.51	65.53
124		0.5	-35.69	62.88	63.47

We see that these values always happen when total output is larger for the best collusive outcome under non bankruptcy. The explanation for this is the following. As we show in the previous example, the possibility of bankruptcy shifts the maximum joint profit that can be supported as an equilibrium increasing the output of the superior firm and decreasing the output of the inferior firm. We have two effects on welfare (as in Lahiri and Ono (1988)). On the one hand, we have the competition effect in which a larger output increases social welfare. On the other hand, we have the technological effect in which the output is produced more inefficiently. Thus, when aggregate output is larger in the best collusive outcome that can be supported with bankruptcy, than the one that can be supported without bankruptcy, both effects go in the same direction and the effect on social welfare is unambiguous (larger under bankruptcy than under non bankruptcy). However, when aggregate output is smaller in the best collusive outcome that can be supported with bankruptcy, than the one that can be supported without bankruptcy, both effects go in different directions and, indeed, the technological effect might overcome the competition effect. Finally, note that this only happens when discount is very close to the minimum discount for which predation is profitable. This is because, for these discount values, predation is not very profitable so the incentive to increase output beyond the Cournot level is small and so is the technological effect. In contrast, when firms do not discount future that much, aggregate output reflect the predation of the superior firm and the technological and the competition effects go in the same direction.

## 5. Final Remarks

Our results are obtained at the cost of making several simplifying assumptions to make the model tractable. Here we discuss some of the issues arising from these simplifications.

### No accumulation

In this paper we focused on outputs that make other firms bankrupt, but we did not consider the funds that might support or deter aggressive strategies (the "deep pocket" argument). Our research strategy is to analyze the incentives to prey in the simplest possible case where no funds can be accumulated. A fully fledged model of accumulation and predation is, no doubt, preferable but it is beyond the scope of our paper. In other cases, accumulation of profits might play an important role in shaping the MPE set as in the model of Rosenthal and Rubinstein (1984).<sup>2</sup>

### Credit

If credit is given on the basis of past performance, the redefinition of the BF set can be applied here and credit can be incorporated into the model. However, if credit is given on the basis of future performance, future performance also depends on credit (via the BF constraints), which makes this problem extremely complex. This points to a deep conceptual problem about credit in oligopolist markets where firms might be made bankrupt. This topic should be the subject of future research.

### Entry

In this paper we assumed that the disappearance of a firm does not bring a new one into the market. Of course this should not be taken literally. What we mean is that if entry does not quickly follow, it makes sense, as a first approximation, to analyze the model with a given number of firms. For instance it can be shown that when firms do not discount future too much and costs and demand are linear, ruining a firm is a good investment even if monopoly lasts for one period. In other cases, though, the nature of equilibria will be altered if, for instance, entry immediately follows the ruin of a competitor as in the model of Rosenthal and Spady (1989).<sup>3</sup>

### Buying Competitors

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<sup>2</sup>They characterize a subset of the Nash equilibria in the repeated game with no discounting (i.e.,  $\delta = 1$ ) where each player regards the ruin of the other player as the best possible outcome and his own ruin as the worst possible outcome.

<sup>3</sup>They consider a prisoner's dilemma in continuous time in a market with room for two firms only. When a firm goes bankrupt, this firm is immediately replaced by a new entrant. They show that some kind of predatory behavior may arise in equilibrium.

In our model, there is no option to buy a firm. Sometimes it is argued that buying an opponent may be a cheaper and safer strategy than ruining it. We do not deny that buying competitors plays an important role in business practices. However, we do not agree that under the option of buying, ruining a competitor is irrational. First, buying competitors may be forbidden by a regulatory body because of anticompetitive effects. Second, when the owner of a firm sells it to competitors, this does not stop her from creating a new firm and financing it with the money received from selling the old one. In other words, selling a firm is not equivalent to a contract in which the owner commits not to enter into a market again. Thus, bankruptcy may be the only credible way of getting rid of a competitor. Finally, buying and ruining competitors may complement each other because the acquisition value may depend on the aggressiveness of the buyer in the past; see Burns (1986) for some evidence in the American tobacco industry. Thus, it seems that a better understanding of the mechanism of ruin might help the further enhancement of our understanding of how the buying mechanism works in this case.

Summing up, the model presented in this paper sheds some light on certain aspects of the equilibrium in oligopolist markets in which firms may make each other bankrupt. We hope that the insights obtained here can be used in further research in this area.

## 6. APPENDIX

**Proof of Proposition 2.** By Lemma 4, the existence of MPE is established for  $\delta \leq \underline{\delta}$ . So, in the following proof, we only consider cases in which  $\delta > \underline{\delta}$ .

There are 4 possible states in our dynamic game. Namely, the set of active firms is (1) S and I, (2) S (3) I, and (4) empty.<sup>4</sup> Since there is a single active firm (monopoly) in states (2) and (3), that firm simply chooses the monopoly output. In (4), no strategic decision can be made. In this way, for any MPE, Markovian strategies in states (2), (3) and (4) are uniquely determined, and continuation profits in those states are derived accordingly. Note also that state transition is irreversible. If a state changes from (1) to any other one from (2) to (4), it is impossible for firms to move this back to (1). Moreover, the continuing game in MPE must stay there forever; no further

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<sup>4</sup>Given our assumption that firm *S* goes bankrupt only if firm *I* goes bankrupt, state (3) never arises. Such property, to some extent, simplifies the following proof, but is not needed to establish the existence result. To clarify this point, we do not impose this assumption in what follows.

transition can occur.

Since equilibrium Markovian strategies in (2), (3) and (4) are derived above, we focus on Markovian strategies in the remaining state (1) where both firms are active. Let  $\sigma_S$  and  $\sigma_I$  be a (mixed) strategy in (1) for  $S$  and  $I$ , respectively. Note that, depending on realized pure actions of  $\sigma_S$  and  $\sigma_I$ , state transition, i.e., bankruptcy, may occur. Let  $p_s(\sigma_S, \sigma_I)$ ,  $s = 1, 2, 3, 4$ , be a probability such that state ( $s$ ) would realize when firms play  $\sigma_S$  and  $\sigma_I$ . Then, continuation profit in state (1) for each firm, denoted by  $V_i$ ,  $i = S, I$ , is expressed as follows.

$$V_S(\sigma_S, \sigma_I) = \pi_S(\sigma_S, \sigma_I) + \delta \left[ p_1(\sigma_S, \sigma_I)V_S(\sigma_S, \sigma_I) + p_2(\sigma_S, \sigma_I)\frac{\pi_S^M}{1-\delta} \right] \quad (6.1)$$

$$V_I(\sigma_S, \sigma_I) = \pi_I(\sigma_S, \sigma_I) + \delta \left[ p_1(\sigma_S, \sigma_I)V_I(\sigma_S, \sigma_I) + p_3(\sigma_S, \sigma_I)\frac{\pi_I^M}{1-\delta} \right] \quad (6.2)$$

where  $\pi_i^M$  is a monopoly profit of firm  $i = S, I$ . Solving each equation, we obtain the following.

$$V_S(\sigma_S, \sigma_I) = \frac{1}{1 - \delta p_1(\sigma_S, \sigma_I)} \left[ \pi_S(\sigma_S, \sigma_I) + \delta p_2(\sigma_S, \sigma_I)\frac{\pi_S^M}{1-\delta} \right] \quad (6.3)$$

$$V_I(\sigma_S, \sigma_I) = \frac{1}{1 - \delta p_1(\sigma_S, \sigma_I)} \left[ \pi_I(\sigma_S, \sigma_I) + \delta p_3(\sigma_S, \sigma_I)\frac{\pi_I^M}{1-\delta} \right] \quad (6.4)$$

Note that  $V_i$ ,  $i = S, I$  is discontinuous in pure actions  $(x_S, x_I)$ , since the probability of state realization,  $p_s$ ,  $s = 1, 2, 3, 4$ , is discontinuous in  $(x_S, x_I)$ . To establish the existence of MPE, it is necessary and sufficient to show that a Markovian strategy profile  $(\sigma_S, \sigma_I)$  constitutes an SPNE among all Markovian strategies.<sup>5</sup> In our dynamic game, deviation in states (2) and (3) cannot be profitable for a remaining active firm, and no deviation is possible in (4). Therefore, without loss of generality, we can focus on deviation in state (1) alone. This implies that an MPE of our dynamic game is identical to a Nash equilibrium of a static game in which the payoff function of each firm is set equal to  $V_i(\sigma_S, \sigma_I)$ ,  $i = S, I$ . We shall denote this (modified) static game by  $G$ .

Fortunately, Theorem 5b in Dasgupta and Maskin (1986) (D&M hereinafter) can be invoked to show the existence of a mixed strategy Nash equilibrium of  $G$ . Roughly speaking, the existence is guaranteed when utility functions are bounded and continuous except in a set of measure zero in the (joint) pure strategy space. More precisely, the theorem requires that (a) discontinuities occur in a set whose dimension is strictly lower than the dimension of the strategy space, (b)

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<sup>5</sup>Necessity is obvious. To understand sufficiency, note that, if the opponent uses Markovian strategy, the player always has a best reply that is Markovian as well. See, for example, pp.501 of Fudenberg and Tirole (1991) for more detailed discussion.

strategy sets are intervals, and (c) when we approach a discontinuity if a firm profit falls, another rises. Our game clearly satisfies (b). Since profit function,  $\pi_i, i = S, I$ , is continuous in  $(x_S, x_I)$ , discontinuity possibly occurs only at a point in which  $p_s, s = 1, 2, 3, 4$ , is discontinuous, or a state transition occurs. Our bankruptcy conditions assure that such state transitions occur in a set whose dimension is strictly lower than the dimension of the strategy space, and hence (a) is also satisfied. Therefore, we only need to check (c).

Given the irreversibility of the state transition mentioned above, there are only 4 possible cases of discontinuities of  $p_s$  (or  $V_i$ ): (i)  $p_1 = 1 \rightarrow p_2 = 1$ , (ii)  $p_1 = 1 \rightarrow p_3 = 1$ , (iii)  $p_2 = 1 \rightarrow p_4 = 1$ , and (iv)  $p_3 = 1 \rightarrow p_4 = 1$ . In (iii), firm  $I$  is not active either before or after the state transition from (2) to (4) occurs. That is,  $V_I = 0$  in both (2) and (4), and hence condition (c) is trivially satisfied. By the symmetric argument, (c) is also satisfied in (iv). In (i) and (ii), when the profit of one firm falls (because this firm goes bankrupt), the profit of the other firm must rise (because the remaining firm becomes a monopolist). So, (c) is satisfied in cases (i) and (ii) as well.

We have shown that  $G$  satisfies all the conditions of D&M, and hence the existence of a mixed strategy Nash equilibrium of  $G$  is guaranteed. Since this equilibrium is essentially identical to the MPE of our dynamic game, the existence of an MPE is also guaranteed. ■

### Characterization of the support of the Markovian mixed strategy equilibrium.

The Proof of Proposition 3 follows from the following lemmas.

**Lemma 7.** *For any mixed strategy of the other firm, the optimal output that maximizes a firm's expected profit in a period (i.e.  $E[\pi_i(x_i, x_j) \mid \sigma_j]$ ) is always unique.*

**Proof.** From Assumption EC,  $\arg \max_{x_i > 0} E[\pi_i(x_i, x_j) \mid \sigma_j]$  is unique (if it exists), and  $i$ 's optimal output is either  $\arg \max_{x_i > 0} E[\pi_i(x_i, x_j) \mid \sigma_j]$  or 0. Assumption AC implies that the expected profit of the former is always positive, so it cannot be the case that both become optimal. ■

**Lemma 8.** *Given  $\delta > \underline{\delta}$ , for any equilibrium in mixed strategies,*

- (i) *at least one firm goes bankrupt with strictly positive probability,*
- (ii) *both firms use totally mixed strategies.*

**Proof.** (i) Since  $\delta > \underline{\delta}$ , at least one firm is using a totally mixed strategy. Suppose without loss of generality that it is firm  $j$  and suppose that no firm goes bankrupt. Pick any two outputs  $x, x'$  from the support of the mixed strategy of firm  $j$ . Then, the firm must be indifferent between

choosing  $x$  and  $x'$ . However, given that no bankruptcy occurs, the firm's optimal output (to the other firm's equilibrium strategy) is always unique by Lemma 7. Thus, we get a contradiction.

(ii) Suppose on the contrary that firm  $i$  uses pure strategy  $x_i$  (and  $j$  uses mixed strategy  $\sigma_j$ ). Then, firm  $j$  does not go bankrupt in equilibrium, since choosing such an output in the support of  $j$  is clearly suboptimal. Let  $x < x'$  be two different outputs in the support of  $\sigma_j$ . Lemma 7 implies that, in order for  $j$  to be indifferent between  $x$  and  $x'$ , bankruptcy must occur in either output. Since  $\pi_i$  is decreasing in  $x_j$ ,  $i$  must go bankrupt under  $x'$  but not under  $x$ . Moreover, the support of  $\sigma_j$  cannot contain a point other than  $x$  and  $x'$  (since choosing such an output cannot yield the same profit as  $x$  and  $x'$  do). However, given that  $j$  randomizes only over the two points  $x$  and  $x'$ ,  $x$  must be the best reply to  $x_i$  in the one period game, since bankruptcy does not occur in such a case. Then,  $j$  has incentive to set  $x'$  as small as possible to make  $i$  go bankrupt. But since the set of outputs that make  $i$  bankrupt is open, such a minimum output does not exist. In this way,  $(x_i, \sigma_j)$  cannot be a mutual best reply. ■

Let us denote the inferior and the superior outputs in the support of the equilibrium (mixed) strategy for firm  $i$  by  $\underline{x}_i$  and  $\overline{x}_i$ , respectively. By Lemma 8 (ii), we have  $\underline{x}_i < \overline{x}_i$  for each  $i$ .

**Lemma 9.** *In equilibrium, the following condition must hold for every firm  $i$ :*

$$\pi_i(\overline{x}_i, \underline{x}_j) > 0.$$

**Proof.** Suppose  $\pi_i(\overline{x}_i, \underline{x}_j) \leq 0$ . By choosing  $x_i = \overline{x}_i$ , firm  $i$  always receives non-positive profit, and strictly negative profit when  $x_j > \underline{x}_j$  (note  $\pi_i(x_i, x_j)$  is decreasing in  $x_j$ ). Since the latter case occurs with positive probability,  $i$  would always become strictly better off by choosing  $x_i = 0$ . ■

**Lemma 10.** *In equilibrium, the following condition must hold for at least one firm:*

$$\pi_i(\underline{x}_i, \overline{x}_j) > 0.$$

**Proof.** Suppose on the contrary that  $\pi_i(\underline{x}_i, \overline{x}_j) \leq 0$  and  $\pi_j(\underline{x}_j, \overline{x}_i) \leq 0$ . Combining with Lemma 9, the following inequalities must hold.

$$\pi_i(\overline{x}_i, \underline{x}_j) > 0 \text{ and } \pi_j(\underline{x}_j, \overline{x}_i) \leq 0 \Rightarrow AC_i(\overline{x}_i) < p(\overline{x}_i + \underline{x}_j) \leq AC_j(\underline{x}_j), \quad (6.5)$$

$$\pi_j(\overline{x}_j, \underline{x}_i) > 0 \text{ and } \pi_i(\underline{x}_i, \overline{x}_j) \leq 0 \Rightarrow AC_j(\overline{x}_j) < p(\overline{x}_j + \underline{x}_i) \leq AC_i(\underline{x}_i). \quad (6.6)$$



Since average costs are constant, the above conditions imply

$$p(\bar{x}_j + \underline{x}_i) \leq AC_i(\underline{x}_i) = AC_i(\bar{x}_i) < p(\bar{x}_i + \underline{x}_j), \quad (6.7)$$

$$p(\bar{x}_i + \underline{x}_j) \leq AC_j(\underline{x}_j) = AC_j(\bar{x}_j) < p(\bar{x}_j + \underline{x}_i), \quad (6.8)$$

which is an obvious contradiction. ■

**Lemma 11.** *If  $\pi_i(\underline{x}_i, \bar{x}_j) > 0$  holds, then*

- (i)  $\underline{x}_i$  maximizes the expected per period profit given  $j$ 's mixed strategy,  $E[\pi_i(x_i, x_j) \mid \sigma_j]$ ,
- (ii)  $\underline{x}_i$  must be isolated from other parts of the support of  $i$ 's equilibrium strategy.

**Proof.** Since  $\pi_j(\bar{x}_j, \underline{x}_i) > 0$  by Lemma 9,  $j$ 's profit  $\pi_j(x_j, \underline{x}_i)$  is strictly positive for any  $x_j \in [0, \bar{x}_j]$ . This implies that probability such that  $j$  goes bankrupt is 0 when  $i$  chooses an output sufficiently close to  $\underline{x}_i$ . Therefore, (i)  $\underline{x}_i$  must be optimal given that no bankruptcy occurs, and (ii) no output close to  $\underline{x}_i$  can be contained in  $i$ 's equilibrium support. ■

**Lemma 12.** *In equilibrium, firm  $I$  never produces an output strictly higher than the one which maximizes its per-period profit given firm  $S$ 's equilibrium mixed strategy. That is,*

$$\bar{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S].$$

**Proof.** Note first that  $I$  can make  $S$  bankrupt only if  $I$  itself goes bankrupt. Hence, firm  $I$  can never be better off by bankrupting firm  $S$ . Choosing  $x_I > \bar{x}_I$  weakly increases the risk of bankruptcy and strictly reduces  $\pi_I$  in that period. Therefore, it must be suboptimal. ■

**Lemma 13.** *The following conditions must hold:*

- (i)  $\pi_I(\underline{x}_I, \bar{x}_S) \leq 0$ .
- (ii)  $\pi_S(\underline{x}_S, \bar{x}_I) > 0$ .

**Proof.** We first verify (i). Suppose on the contrary that  $\pi_I(\underline{x}_I, \bar{x}_S) > 0$  holds. Then, by Lemma 11,  $I$  must choose a strictly larger output than  $\arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ , which contradicts Lemma 12. Given that (i) holds, (ii) must be satisfied by Lemma 10. ■

**Lemma 14.** *Let  $x_S^0 = \arg \max_{x_S} E[\pi_S(x_S, x_I) \mid \sigma_I]$ . The equilibrium support of firm  $S$ 's mixed strategy is such that  $\underline{x}_S = x_S^0$ .*

**Proof.** By Lemma 13,  $\pi_S(\underline{x}_S, \overline{x}_I) > 0$ . Thus, by Lemma 11,  $\underline{x}_S$  maximizes  $S$ 's expected profit (per period) given  $I$ 's mixed strategy and it is an isolated point. Therefore,  $\underline{x}_S = x_S^0$ .

**Lemma 15.** *In equilibrium, firm  $I$  goes bankrupt with positive probability.*

**Proof.** By Lemma 8 at least one firm goes bankrupt. If  $I$  does not go bankrupt then firm  $S$  is bankrupt, but this is impossible because whenever firm  $S$  is bankrupt firm  $I$  is also bankrupt. ■

**Lemma 16.** *For all  $x_I$  in  $I$ 's mixed strategy support,  $\pi_I(x_I, \overline{x}_S) \leq 0$ .*

**Proof.** This follows immediately from Lemma 13. ■

**Lemma 17.** *The probability of bankruptcy at the infimum of the support of  $I$ 's mixed strategy,  $\underline{x}_I$ , is zero. There is no other output in the support with zero probability of bankruptcy.*

**Proof.** Suppose on the contrary that the probability that  $I$  goes bankrupt at  $\underline{x}_I$  is positive. This probability is the probability that the superior firm produces  $x_S \in (\tilde{x}_S, \overline{x}_S]$ , for  $\tilde{x}_S$  such that  $p(\underline{x}_I + \tilde{x}_S) = c_I$ . By Lemma 9 we know that  $\pi_I(\overline{x}_I, \underline{x}_S) > 0$ , consequently,  $\pi_I(\underline{x}_I, \underline{x}_S) > 0$ . Thus,  $\tilde{x}_S \in (\underline{x}_S, \overline{x}_S]$ . If  $\tilde{x}_S$  is in the support of  $S$ 's mixed strategy, firm  $S$ , by concentrating all the mass placed at  $[\tilde{x}_S + \varepsilon, \overline{x}_S]$  in  $\tilde{x}_S + \varepsilon$ , the probability of bankruptcy will not change for firm  $I$  and will increase the per period profit of firm  $S$ . If  $\tilde{x}_S$  is not in the support of  $S$ 's mixed strategy, let  $\hat{x}_S > \tilde{x}_S$  be the closest point to  $\tilde{x}_S$  in the support of  $S$ 's mixed strategy. Again, firm  $S$ , by placing all the mass placed at  $[\hat{x}_S, \overline{x}_S]$  in  $\hat{x}_S$ , will not change the probability of bankruptcy for firm  $I$  and will increase its per period profit. Finally, by Lemma 12,  $\overline{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ . If there is another output in the support different from  $\overline{x}_I$  with zero probability of bankruptcy,  $\tilde{x}_I$ , should be such that  $\underline{x}_I < \tilde{x}_I \leq \overline{x}_I$ . But then, since firm  $i$ 's expected profit function (given the other firm's mixed strategy) is strictly concave in  $x_i$ , it could not be that the profit of firm  $I$  is the same at  $\underline{x}_I$  and at  $\tilde{x}_I$ . ■

**Lemma 18.** *The support of  $I$ 's mixed strategy contains at least an interval and the infimum of the interval is  $\underline{x}_I$ .*

**Proof.** Let us see first that  $\underline{x}_I$  cannot be an isolated mass point. If it were,  $\underline{x}_I$  would be the minimum of the support of  $I$ 's mixed strategy. By Lemma 17 the probability of bankruptcy at  $\underline{x}_I$

is zero. Thus,  $\underline{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ , but this contradicts Lemma 12. Thus,  $\underline{x}_I$  is not an isolated point and, for the same argument as before,  $\underline{x}_I$  cannot be in the support of  $I$ 's mixed strategy. ■

**Lemma 19.** *The equilibrium support of firm  $S$ 's mixed strategy contains at least one interval.*

**Proof.** By Lemma 18 the support of  $I$ 's mixed strategy contains at least one interval, and since  $\bar{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ ,  $I$ 's per period profit is strictly increasing and also continuous at any  $x_I$  in the interval. Thus, the probability of bankruptcy must be increasing with  $x_I$  (in order for  $I$  to be indifferent in the interval). This is possible only when the support of  $S$ 's mixed strategy also contains an interval where the distribution of  $x_S$  does not make any jump. ■

**Lemma 20.** *The support of  $I$ 's mixed strategy contains a unique interval and no isolated points.*

**Proof.** Suppose on the contrary that it contains two intervals  $(\underline{x}_I^1, \bar{x}_I^1)$  and  $(\underline{x}_I^2, \bar{x}_I^2)$  with  $\underline{x}_I^1 < \bar{x}_I^1 < \underline{x}_I^2 < \bar{x}_I^2$ , and that there are no isolated outputs between the intervals. Let  $x_S(\bar{x}_I^1)$  and  $x_S(\bar{x}_I^2)$  be such that  $\pi_I(\bar{x}_I^1, x_S(\bar{x}_I^1)) = \pi_I(\bar{x}_I^2, x_S(\bar{x}_I^2)) = 0$ . Note first that it is not optimal for firm  $S$  to have in the support of its mixed strategy outputs in  $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$ , because otherwise firm  $S$ , by concentrating all the mass of the interval  $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$  in  $x_S(\bar{x}_I^2)$ , will not change the probability of bankruptcy of firm  $I$  and will increase its per period profit. If there are isolated points between the two intervals, we can replicate the same argument by considering the biggest isolated point and the greatest interval. Without loss of generality we can consider just the two intervals without isolated points in the middle. Let  $x_I = \bar{x}_I^1 + \varepsilon$  with  $\varepsilon$  sufficiently small. Since there is no mass in  $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$ , the probability of bankruptcy at  $\bar{x}_I^1 + \varepsilon$  is the same as at  $\bar{x}_I^1$  but the per period payoff is bigger. So it cannot be optimal for firm  $I$  to produce  $\bar{x}_I^1$ . Thus, there is only one interval in the support of  $I$ 's mixed strategy. The same argument can be applied in order to discard the isolated points in the support of  $I$ 's mixed strategy. ■

**Lemma 21.** *At equilibrium the following should hold:*

- (i)  $\pi_I(\underline{x}_I, \bar{x}_S) = 0$ ;
- (ii)  $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$ , where  $\underline{x}_S^1$  is the infimum of the first interval in the support of  $S$ 's mixed strategy.

**Proof.**

- (i) Lemma 17 implies  $\pi_I(\underline{x}_I, \bar{x}_S) \geq 0$ . By Lemma 13  $\pi_I(\underline{x}_I, \bar{x}_S) \leq 0$ . Thus,  $\pi_I(\underline{x}_I, \bar{x}_S) = 0$ .

(ii) Suppose that  $\pi_I(\bar{x}_I, \underline{x}_S^1) < 0$ . By Lemmas 9 and 14,  $\pi_I(\bar{x}_I, x_S^0) > 0$ . Then,  $x_S(\bar{x}_I)$  defined by  $\pi_I(\bar{x}_I, x_S(\bar{x}_I)) = 0$  is such that  $x_S^0 < x_S(\bar{x}_I) < \underline{x}_S^1$ . This implies that when  $\varepsilon$  is sufficiently small, the probability of bankruptcy at any  $x_I \in (\bar{x}_I - \varepsilon, \bar{x}_I)$  does not change. But then,  $I$  cannot be indifferent between any two outputs in the interval  $(\bar{x}_I - \varepsilon, \bar{x}_I)$ . Thus,  $\pi_I(\bar{x}_I, \underline{x}_S^1) \geq 0$ . Suppose that  $\pi_I(\bar{x}_I, \underline{x}_S^1) > 0$ , then, at any  $x_S \in (\underline{x}_S^1, x_S(\bar{x}_I))$ , the probability of bankruptcy for  $I$  is the same, but then, firm  $S$  cannot be indifferent in the interval  $(\underline{x}_S^1, \underline{x}_S^1 + \varepsilon)$ . Thus,  $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$ . ■

**Lemma 22.** *At equilibrium,  $\bar{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ .*

**Proof.** Suppose that  $\bar{x}_I < \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ . By Lemmas 9 and 14,  $\pi_I(\bar{x}_I, x_S^0) > 0$  and by Lemma 21,  $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$ . At an output  $x_I = \bar{x}_I + \varepsilon$  with  $\varepsilon$  sufficiently small such that  $\bar{x}_I + \varepsilon < \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ , the probability of bankruptcy is the same as at  $\bar{x}_I$ , but the per period payoff is greater. So  $\bar{x}_I$  cannot be the best reply to  $\sigma_S$ . Thus,  $\bar{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ . ■

**Lemma 23.** *The support of  $S$ 's mixed strategy contains a unique interval and one isolated point.*

**Proof.** By Lemma 14  $\underline{x}_S = x_S^0$  and is isolated. By Lemma 21, more than one interval in the support of  $S$ 's mixed strategy would imply that some outputs in the support of  $I$ 's mixed strategy produce the same probability of bankruptcy for firm  $I$ . But, if this is the case, firm  $I$  cannot be indifferent among those outputs. The same kind of argument can be applied to discard any isolated point bigger than the supremum of the interval. Finally, given that  $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$ , if there are two isolated points smaller than the infimum of the interval then none of those outputs can bankrupt firm  $I$ , but then firm  $S$  cannot be indifferent among them. ■

**Lemma 24.** *The infimum of the interval in the support of firm  $S$ 's mixed strategy is not in the support.*

**Proof.** Let  $\underline{x}_S^*$  the infimum of the interval in the support. Since by Lemma 21  $\pi_I(\bar{x}_I, \underline{x}_S^*) = 0$ , by producing  $\underline{x}_S^*$  the probability of bankruptcy for firm  $I$  is zero. But this probability is also zero by producing  $x_S^0$  (the isolated point in  $S$ 's support which is the minimum output in the support). But then, firm  $S$  cannot be indifferent between  $x_S^0$  and  $\underline{x}_S^*$ . ■

**Lemma 25.** *There is a mass point at the supremum of the support of  $I$ 's mixed strategy,  $\bar{x}_I$ .*

**Proof.** Suppose not. Given that  $\pi_I(\overline{x_I}, \underline{x_S}^*) = 0$ , the probability of bankruptcy for firm  $I$  when  $x_S \rightarrow \underline{x_S}^*$  is zero. But this is not possible because at  $x_S^0$  the probability of bankruptcy for firm  $I$  is also zero and  $S$  is better off at  $x_S^0$ . Therefore, there is a mass point in  $\overline{x_I}, p_I$ . In this case, the probability of bankruptcy for firm  $I$  when  $x_S \rightarrow \underline{x_S}^*$  is  $p_I$  but at  $x_S^0$  is zero ■

Those Lemmas complete the proof of Proposition 3.

#### Proof of Proposition 4

We start by showing that if the demand is linear, (v) and (vi) in Proposition 3 implies that  $\overline{x_I}^* = a_I - \underline{x_S}^*$  and  $\underline{x_I}^* = a_I - \overline{x_S}^*$ . If  $\underline{x_I}^*$  is greater than zero, the implication is direct. Suppose that  $\underline{x_I}^* = 0$  and  $a_I \neq \overline{x_S}^*$ . Suppose that  $\overline{x_S}^* > a_I$ . At any  $x_S$  in a neighborhood of  $\overline{x_S}^*$ , the probability of bankruptcy for firm  $I$  is 1, but then firm  $S$  cannot be indifferent among all the  $x_S$  in that neighborhood. Suppose that  $\overline{x_S}^* < a_I$ , and let  $\tilde{x_I} \in (0, \overline{x_I}^*)$  be such that  $\tilde{x_I} = a_I - \overline{x_S}^*$ , since  $\pi_I(\overline{x_I}^*, \underline{x_S}^*) = 0$ ,  $\tilde{x_I}$  exists and is greater than zero. At any  $x_I \in (0, \tilde{x_I})$  the probability that firm  $I$  is bankrupt is zero, but then firm  $I$  cannot be indifferent among all  $x_I \in (0, \tilde{x_I})$ .

Let  $(\sigma_I, \sigma_S)$  a mixed strategy profile. The payoffs for each firm given  $(\sigma_I, \sigma_S)$  are:

$$\begin{aligned} V_I(\sigma_I, \sigma_S) &= \frac{1}{1 - \delta p_1} \pi_I(\sigma_I, \sigma_S); \\ V_S(\sigma_I, \sigma_S) &= \frac{1}{1 - \delta p_1} \left[ \pi_S(\sigma_I, \sigma_S) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right] \end{aligned}$$

where  $p_1$  is the probability that both firms survive given that strategy and  $p_2$  is the probability that only  $S$  survives.

Let us first prove (i). For firm  $I$ , every  $x_I$  in the support of  $\sigma_I$  has to be the best reply to  $\sigma_S$ . Thus for each  $x_I$  in the support,

$$\frac{\partial V_I(x_I, \sigma_S)}{\partial x_I} = 0,$$

or equivalently

$$\frac{\delta \frac{\partial p_1(x_I, \sigma_S)}{\partial x_I}}{(1 - \delta p_1)^2} \pi_I(x_I, \sigma_S) + \frac{1}{1 - \delta p_1} \frac{\partial \pi_I(x_I, \sigma_S)}{\partial x_I} = 0 \quad (6.9)$$

Given  $(x_I, \sigma_S)$ , let us compute first the probability that both firms survive,  $p_1(x_I, \sigma_S)$ . By the characterization of the support in Proposition 3,  $\pi_I(\underline{x_I}^*, \overline{x_S}^*) = 0$  and  $\pi_I(\overline{x_I}^*, \underline{x_S}^*) = 0$ . This implies that for each  $x_I$  in the support of  $I$ , the infimum output that bankrupt firm  $I$ ,  $(a_I - x_I)$ , is in the

support of firm  $S$  mixed strategy. Thus,

$$\begin{aligned} p_1(x_I, \sigma_S) &= P(x_S \leq a_I - x_I) = p_S + (1 - p_S) \int_{\underline{x}_S^*}^{a_I - x_I} F'_S(x_S) dx_S, \text{ and} \\ \frac{\partial p_1(x_I, \sigma_S)}{\partial x_I} &= -(1 - p_S) F'_S(a_I - x_I). \end{aligned}$$

Furthermore,

$$\frac{\partial \pi_I(x_I, \sigma_S)}{\partial x_I} = (a_I - 2x_I - Ex_S)$$

where  $Ex_S$  is the expected value of the outputs in  $\sigma_S$  support. Let  $F(x_S) = (1 - p_S)F_S(x_S)$ . Thus, expression (6.9) can be written as

$$\frac{-\delta F'(a_I - x_I)}{(1 - \delta p_1)^2} \pi_I(x_I, \sigma_S) + \frac{1}{1 - \delta p_1} \frac{\partial \pi_I(x_I, \sigma_S)}{\partial x_I} = 0,$$

or equivalently

$$\frac{-\delta F'(a_I - x_I)}{1 - \delta p_1} (a_I - x_I - Ex_S)x_I + (a_I - 2x_I - Ex_S) = 0 \quad (6.10)$$

Given that for each  $x_I$ ,  $a_I - x_I \in (\underline{x}_S^*, \overline{x}_S^*)$ , and for each  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$ , there is  $x_I \in (\underline{x}_I^*, \overline{x}_I^*)$  such that  $a_I - x_I = x_S$ , (6.10) can be expressed in terms of  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$ ,

$$\frac{-\delta F'(x_S)}{1 - \delta p_S - \delta F(x_S)} (x_S - Ex_S)(a_I - x_S) + (2x_S - a_I - Ex_S) = 0 \quad (6.11)$$

Let  $H(x_S) = 1 - \delta p_S - \delta F(x_S)$ , and  $f(x_S) = (x_S - Ex_S)(a_I - x_S)$ . Note that  $H'(x_S) = -\delta F'(x_S)$ , and  $f'(x_S) = a_I - x_S - x_S + Ex_S = -(2x_S - a_I - Ex_S)$ . Using this transformation, (6.11) can be written as:

$$\frac{H'(x_S)}{H(x_S)} f(x_S) - f'(x_S) = 0$$

Thus, for  $H(x_S) = Kf(x_S)$  the above equation holds, which implies that

$$1 - \delta p_S - \delta F(x_S) = K(x_S - Ex_S)(a_I - x_S), \text{ and} \quad (6.12)$$

$$\delta F'(x_S) = K(2x_S - a_I - Ex_S) \quad (6.13)$$

By (iii) in Proposition 3,  $\overline{x}_I^* = \arg \max_{x_I} \pi_I(x_I, \sigma_S)$ . Therefore,  $\overline{x}_I^*$  has to be the solution of the first order condition

$$a_I - 2\overline{x}_I^* - Ex_S = 0.$$

By point (vi) in Proposition 3,  $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$ , which implies that  $\overline{x}_I^* = a_I - \underline{x}_S^*$ . Thus,

$$Ex_S = 2\underline{x}_S^* - a_I \quad (6.14)$$

Which implies that

$$\delta F'(x_S) = 2K(x_S - \underline{x}_S^*) \quad (6.15)$$

Furthermore,

$$1 - p_S = \int_{\underline{x}_S^*}^{\overline{x}_S^*} F'(x_S) dx_S.$$

Using (6.15) and integrating,

$$1 - p_S = \frac{K}{\delta} (\overline{x}_S^* - \underline{x}_S^*)^2 \quad (6.16)$$

By (6.15), (6.16) and given that  $(1 - p_S)F'_S(x_S) = F'(x_S)$ ,

$$F'_S(x_S) = \frac{2(x_S - \underline{x}_S^*)}{(\overline{x}_S^* - \underline{x}_S^*)^2}, \quad (6.17)$$

which proves the first part of (i). For the second part, substituting  $p_S$  and  $Ex_S$  in (6.12) using (6.14) and (6.16) we get

$$\begin{aligned} 1 - \delta + K(\overline{x}_S^* - \underline{x}_S^*)^2 - \delta F(x_S) &= K(x_S - 2\underline{x}_S^* + a_I)(a_I - x_S) \\ F(x_S) &= \frac{1}{\delta} - 1 + \frac{K}{\delta} (\overline{x}_S^* - \underline{x}_S^*)^2 - \frac{K}{\delta} (x_S - 2\underline{x}_S^* + a_I)(a_I - x_S) \end{aligned}$$

Given that  $F(\overline{x}_S^*) = \frac{K}{\delta} (\overline{x}_S^* - \underline{x}_S^*)^2$ ,

$$\frac{1}{\delta} - 1 - \frac{K}{\delta} (\overline{x}_S^* - 2\underline{x}_S^* + a_I)(a_I - \overline{x}_S^*) = 0.$$

Which implies that  $K$  should be

$$K = \frac{1 - \delta}{(\overline{x}_S^* - 2\underline{x}_S^* + a_I)(a_I - \overline{x}_S^*)}$$

Note that  $K$  is positive because  $a_I - \overline{x}_S^* = \underline{x}_I^*$ , and  $a_I - \underline{x}_S^* = \overline{x}_I^*$ . By (6.16)

$$p_S = 1 - F(\overline{x}_S^*) = 1 - \frac{(1 - \delta)(\overline{x}_S^* - \underline{x}_S^*)^2}{\delta(\overline{x}_S^* - 2\underline{x}_S^* + a_I)(a_I - \overline{x}_S^*)} \quad (6.18)$$

Secondly, we prove (ii). For firm  $S$ , every  $x_S$  in the support of  $\sigma_S$  has to be a best reply to  $\sigma_I$ . In particular, for each  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$

$$\frac{\partial V_S(x_S, \sigma_I)}{\partial x_S} = 0,$$

or equivalently,

$$\frac{\delta \frac{\partial p_1(x_S, \sigma_I)}{\partial x_S}}{(1 - \delta p_1)} \left[ \pi_S(x_S, \sigma_I) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right] + \left[ \frac{\partial \pi_S(x_S, \sigma_I)}{\partial x_S} + \delta \frac{\pi_S^M}{1 - \delta} \frac{\partial p_2(x_S, \sigma_I)}{\partial x_S} \right] = 0 \quad (6.19)$$

By the (v) and (vi) in Proposition 3,  $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$  and  $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$ . This implies that for each  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$  in the support of  $S$ , the infimum output for firm  $I$  that bankrupt firm  $I$  is  $x_I = a_I - x_S$ , and  $a_I - x_S \in (\underline{x}_I^*, \overline{x}_I^*)$ . Thus, for  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$  the probability that both firms survive,  $p_1(x_S, \sigma_I)$  is

$$p_1(x_S, \sigma_I) = P(x_I \leq a_I - x_S) = (1 - p_I) \int_{\underline{x}_I^*}^{a_I - x_S} G'_I(x_I) dx_I, \text{ and} \quad (6.20)$$

$$\frac{\partial p_1(x_S, \sigma_I)}{\partial x_S} = -(1 - p_I) G'_I(a_I - x_S). \quad (6.21)$$

Given  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$  the probability that only firm  $S$  survives (or equivalently, the probability that firm  $S$  bankrupts firm  $I$ ) is

$$p_2(x_S, \sigma_I) = P(x_I > a_I - x_S) = p_I + (1 - p_I) \int_{a_I - x_S}^{\overline{x}_I^*} G'_I(x_I) dx_I \text{ and} \quad (6.22)$$

$$\frac{\partial p_2(x_S, \sigma_I)}{\partial x_S} = (1 - p_I) G'_I(a_I - x_S) \quad (6.23)$$

Let  $G(x_I) = (1 - p_I) G_I(x_I)$ . Plugging (6.20), (6.21), (6.22) and (6.23) into (6.19) we get

$$\frac{-\delta G'(a_I - x_S)}{1 - \delta G(a_I - x_S)} \left[ \pi_S(x_S, \sigma_I) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right] + \left[ \frac{\partial \pi_S(x_S, \sigma_I)}{\partial x_S} + \delta \frac{\pi_S^M}{1 - \delta} G'(a_I - x_S) \right] = 0, \quad (6.24)$$

For each  $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$ ,

$$\begin{aligned} \pi_S(x_S, \sigma_I) &= (a_S - x_S - Ex_I) x_S = (a_S - a_I + x_I - Ex_I)(a_I - x_I) \\ \frac{\partial \pi_S(x_S, \sigma_I)}{\partial x_S} &= (a_S - 2x_S - Ex_I) = (a_S - 2(a_I - x_I) - Ex_I), \end{aligned}$$

where  $Ex_I$  is the expected value of the outputs in  $\sigma_I$  support.

Given (6.22) and the transformation  $G(x_S) = (1 - p_I) G_I(x_S)$ ,  $p_2(x_S, \sigma_I) = 1 - G(a_I - x_S)$ . By considering the terms in (6.24) that multiply  $\pi_S^M$  and using that  $p_2(x_S, \sigma_I) = 1 - G(a_I - x_S)$ ,

$$\delta \frac{\pi_S^M}{1 - \delta} G'(a_I - x_S) \left[ 1 + \frac{-\delta + \delta G(a_I - x_S)}{1 - \delta G(a_I - x_S)} \right] = \frac{\delta \pi_S^M G'(a_I - x_S)}{1 - \delta G(a_I - x_S)}.$$



Thus, equation (6.24) can be expressed in terms of  $x_I \in (\underline{x}_I^*, \overline{x}_I^*)$  as

$$\frac{-\delta G'(x_I)}{1 - \delta G(x_I)} [(a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M] + (a_S - 2(a_I - x_I) - Ex_I) = 0. \quad (6.25)$$

Let  $J(x_I) = 1 - \delta G(x_I)$  and  $g(x_I) = (a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M$ . Note that  $J'(x_I) = -\delta G'(x_I)$  and  $g'(x_I) = -(a_S - 2(a_I - x_I) - Ex_I)$ . Using this transformation, (6.25) can be written as:

$$\frac{J'(x_I)}{J(x_I)} g(x_I) - g'(x_I) = 0$$

Thus, for  $J(x_I) = Bg(x_I)$  the above equation will hold, which implies that

$$\begin{aligned} 1 - \delta G(x_I) &= B [(a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M] \\ -\delta G'(x_I) &= B(2(a_I - x_I) - a_S + Ex_I) \end{aligned}$$

By (iv) in Proposition 3,  $x_S^0 = \arg \max_{x_S} \pi_S(x_S, \sigma_I)$ . Therefore,  $x_S^0$  has to be the solution of the first order condition  $a_S - 2x_S^0 - Ex_I = 0$ . Thus,

$$Ex_I = a_S - 2x_S^0 \quad (6.26)$$

Which implies that

$$G(x_I) = \frac{1}{\delta} - \frac{B}{\delta} [(2x_S^0 - a_I + x_I)(a_I - x_I) - \pi_S^M], \text{ and} \quad (6.27)$$

$$\delta G'(x_I) = B(2(x_S^0 - (a_I - x_I))) \quad (6.28)$$

Using (6.28) and integrating,

$$1 - p_I = \int_{\underline{x}_I^*}^{\overline{x}_I^*} G'(x_I) dx_I = \frac{B}{\delta} [(a_I - \overline{x}_I^*)^2 - (a_I - \underline{x}_I^*)^2 + 2x_S^0(\overline{x}_I^* - \underline{x}_I^*)] = \quad (6.29)$$

$$= \frac{B}{\delta} (\overline{x}_I^* - \underline{x}_I^*) (\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0) \quad (6.30)$$

Since  $G'_I(x_I) = G'(x_I)/(1 - p_I)$ , then

$$G'_I(x_I) = \frac{2(x_S^0 - (a_I - x_I))}{(\overline{x}_I^* - \underline{x}_I^*) (\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0)} \quad (6.31)$$

which proves the first part of (ii). For the second part, since  $G(\underline{x}_I^*) = 0$ , and by (6.27)  $G(\underline{x}_I^*) = \frac{1}{\delta} - \frac{B}{\delta} [(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M]$ ,

$$B = \frac{1}{(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M}$$

which implies that

$$p_I = 1 - \frac{(\overline{x}_I^* - \underline{x}_I^*) \left( \overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0 \right)}{\delta \left[ (2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M \right]}. \quad (6.32)$$

Finally, to prove (iii), (4.6) follows from (6.14) (that is,  $Ex_S = 2x_S^* - a_I$ ); equation (4.7) follows from (6.26) (that is,  $Ex_I = a_S - 2x_S^0$ ); equation (4.8) is obtained because if  $x_S^0$  and  $\overline{x}_S^*$  are in the support of  $\sigma_S$ ,  $V_S(x_S^0, \sigma_I) = V_S(\overline{x}_S^*, \sigma_I)$ . Since by (vii) in Proposition 3  $\pi_I(\overline{x}_I^*, x_S^0) > 0$ , the probability that both firms survive,  $p_1(x_S^0, \sigma_I) = 1$ , and the probability that only firm  $S$  survives,  $p_2(x_S^0, \sigma_I) = 0$ . By (v) in Proposition 3,  $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$ . Thus, at  $(\overline{x}_S^*, \sigma_I)$ ,  $p_1(\overline{x}_S^*, \sigma_I) = 0$ , and the probability that only firm  $S$  survives,  $p_2(\overline{x}_S^*, \sigma_I) = 1$ . Therefore,

$$\begin{aligned} V_S(x_S^0, \sigma_I) &= \frac{1}{1-\delta} \pi_S(x_S^0, \sigma_I); \\ V_S(\overline{x}_S^*, \sigma_I) &= \pi_S(\overline{x}_S^*, \sigma_I) + \delta \frac{\pi_S^M}{1-\delta}. \end{aligned}$$

Since any output in the support of  $\sigma_S$  has to give the same payoff,

$$\begin{aligned} \frac{1}{1-\delta} \pi_S(x_S^0, \sigma_I) &= \pi_S(\overline{x}_S^*, \sigma_I) + \delta \frac{\pi_S^M}{1-\delta} \\ \frac{1}{1-\delta} (a_S - x_S^0 - Ex_I) x_S^0 &= (a_S - \overline{x}_S^* - Ex_I) \overline{x}_S^* + \delta \frac{\pi_S^M}{1-\delta} \end{aligned}$$

By (6.28),  $Ex_I = a_S - 2x_S^0$ . Thus

$$\frac{1}{1-\delta} (x_S^0)^2 = (2x_S^0 - \overline{x}_S^*) \overline{x}_S^* + \delta \frac{\pi_S^M}{1-\delta}.$$

Finally, (4.9) and (4.10) follow from (v) and (vi) in Proposition 3 and our remark at the beginning of this proof. ■

## References

- [1] Bernanke B., and M. Gertler (1989). "Agency Costs, Net Worth, and Business Fluctuations". *The American Economic Review*, 79, 1, 14-31.
- [2] Bolton P., and D. Scharfstein (1990). "A Theory of Predation Based on Agency Problems in Financial Contracting". *American Economic Review*, 80, 93-106.
- [3] Burns M. R. (1986). "Predatory Pricing and the Acquisition Cost of Competitors". *The Journal of Political Economy*, 94, 2, 266-296.

- [4] Cabral L., and M. Riordan (1994). "The Learning Curve, Market Dominance, and Predatory Pricing". *Econometrica*, 62, 1115-1140.
- [5] Couwenberg C. (2001). "Survival Rates in Bankruptcy Systems: Overlooking the Evidence". *Research School Systems, Organisation and Management*, 115. Groningen University.
- [6] Dasgupta P., and E. Maskin (1986). "The existence of Equilibrium in Discontinuous Economic Games, I: Theory". *Review of Economic Studies*, 53: 1-26.
- [7] Fudenberg D., and J. Tirole (1991). *Game Theory*, MIT Press, Cambridge, MA.
- [8] Kawakami T., and Y. Yoshida (1997), "Collusion under Financial Constraints: Collusion or predation when the discount factor is near one?" *Economics Letters*, 54: 175-178.
- [9] Kiyotaki N., and J. Moore (1997). "Credit Cycles". *The Journal of Political Economy*, 105, 2, 211-248.
- [10] Lahiri, S. and Y. Ono (1988). "Helping minor firms reduces welfare". *Economic Journal*, December 1199-1203.
- [11] Milgrom, P. and J. Roberts (1982). "Predation, Reputation, and Entry Deterrence". *Journal of Economic Theory*, 27, 280-312.
- [12] Rosenthal R., and A. Rubinstein (1984). "Repeated Two-Players Games with Ruin". *International Journal of Game Theory*, 13, 3, 155-177.
- [13] Rosenthal R., and R. H. Spady (1989). "Duopoly with Both Ruin and Entry". *Canadian Journal of Economics*, 22, 4, 834-851.
- [14] Roth D. (1996). "Rationalizable Predatory Pricing". *Journal of Economic Theory*, 68, 380-396.
- [15] Spagnolo G. (2000), "Stock-Related Compensation and Product-Market Competition". *RAND Journal of Economics*, 31: 22-42.